



A class of curves in every knot type where chords of high distortion are common

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Abstract

The distortion of a curve is the supremum, taken over distinct pairs of points of the curve, of the ratio of arclength to spatial distance between the points. Gromov asked in 1981 whether a curve in every knot type can be constructed with distortion less than a universal constant C . Answering Gromov's question seems to require the construction of lower bounds on the distortion of knots in terms of some topological invariant. We attempt to make such bounds easier to construct by showing that pairs of points with high distortion are very common on curves of minimum length in the set of curves in a given knot type with distortion bounded above and distortion thickness bounded below.

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1. A new point of view on Gromov's distortion

In [4] Gromov defines the distortion of a continuous curve $\gamma : [a, b] \rightarrow \mathbb{R}^3$ as

$$\delta(\gamma) := \sup_{s \neq t} \frac{d(\gamma(s), \gamma(t); \gamma)}{d(\gamma(s), \gamma(t); \mathbb{R}^3)},$$

where $d(a, b; X)$ is the distance between a and b in the metric space X . We call the fraction inside the supremum the distortion quotient of the pair (s, t) , written $dq_\gamma(s, t)$. The distortion quotient becomes large when points that are some distance apart along γ are close in space. For example, by adding a twist into a knot we can cause the distortion to grow arbitrarily large as seen in Fig. 1. It follows that for any given knot type $[\gamma]$ and any given constant M , there exists a knot $\gamma \in [\gamma]$ so that $\delta(\gamma) > M$. Attempting to find the infimal distortion of a knot type is far more challenging. The distortion of a knot type, $\delta([\gamma]) := \inf_{\gamma \in [\gamma]} \delta(\gamma)$, is clearly bounded below by 1 for all knots, but finding the exact value of $\delta([\gamma])$ for any given knot type is very difficult. In fact, $\delta([\gamma])$ has only been computed when $[\gamma]$ is the unknot. As stated in [4] and proved in [6] the distortion of the unknot is $\pi/2$ and this value is achieved by the round circle. Gromov asks in [4] whether or not there exists a universal upper bound on $\delta([\gamma])$. Though it has attracted considerable interest, Gromov's question has proved very difficult to answer.

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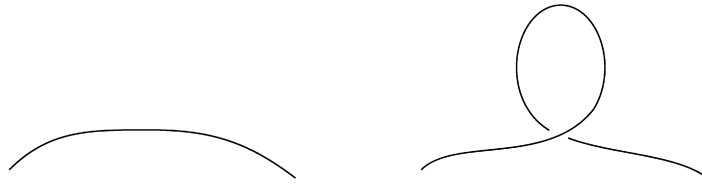


Fig. 1. There can be no upper bound on $\delta([\gamma])$ for any knot type $[\gamma]$. Given any arc of a curve γ of length L , as seen on the left, we can add a twist, as seen on the right. If $\gamma(s)$ lies directly over $\gamma(t)$ in this projection, then given any $\varepsilon > 0$ we can force $d(\gamma(s), \gamma(t); \mathbb{R}^3) < \varepsilon$ while $d(\gamma(s), \gamma(t); \gamma)$ remains constant.

It is natural to conjecture that no such bound exists. A proof would require two steps. We first need to find a topological invariant \mathcal{X} that increases with knot complexity. We must then construct a lower bound on the distortion of a curve γ in terms of \mathcal{X} . Using this lower bound, we can then exhibit a family of knots $\{\gamma_i\}$ so that the sequence $\{\delta([\gamma_i])\}$ diverges. Neither step seems easy to carry out.

Choosing a suitable invariant \mathcal{X} requires care. It is known that a candidate topological invariant must increase without bound for some, but not all, families of knots: for a smooth knot γ there exists an upper bound for the sequence $\{\delta([\gamma]), \delta([\gamma \# \gamma]), \delta([\gamma \# \gamma \# \gamma]), \dots\}$.¹ It follows that if the sequence $\{\mathcal{X}([\gamma]), \mathcal{X}([\gamma \# \gamma]), \mathcal{X}([\gamma \# \gamma \# \gamma]), \dots\}$ diverges, then \mathcal{X} will be of no use when defining a divergent sequence of lower bounds for distortion. This rules out many well-known invariants such as crossing number, bridge number, and genus.

The next task involves constructing a lower bound on the distortion of a curve in terms of the chosen topological invariant. This requires finding points $\gamma(s)$ and $\gamma(t)$ on the curve γ for which $dq_\gamma(s, t)$ is large. Unfortunately, such points can be quite elusive.

The main theorem of this paper, which summarizes the results of [9], shows that length minimizing curves in a particular subset $U_C([\gamma])$ of $[\gamma]$ have the property that points with relatively large distortion quotient saturate arcs of positive total curvature. This should make it easier to bound the distortion of curves in $U_C([\gamma])$ in terms of a topological invariant $\mathcal{X}([\gamma])$.

The set $U_C([\gamma])$ has several membership requirements in addition to the fact that $U_C([\gamma]) \subset [\gamma]$. For technical reasons we assume all curves in $U_C([\gamma])$ have finite total curvature. Second, we require that a constant C bounds the distortion of knots in $U_C([\gamma])$ from above. If we choose C to be a constant multiple of $\delta([\gamma])$, this bound will allow us to relate lower bounds for distortion of curves in $U_C([\gamma])$ to lower bounds on $\delta([\gamma])$. Hence any divergent sequence of lower bounds $\{b_i\}$ on the distortion of knots in, say, $\{U_{2\delta([\gamma_i])}([\gamma_i])\}$ will provide a negative answer to Gromov's question. This is illustrated in Fig. 2.

The last requirement for membership in $U_C([\gamma])$ stems from our interest in studying curves of minimum length. Since distortion is scale invariant the set of length minimizers in $U_C([\gamma])$ will be empty unless we add another constraint. Hence we must fix a scale for curves in $U_C([\gamma])$ to prevent any sequence of curves $\{\gamma_k\} \subset U_C([\gamma])$ approaching a curve γ_0 of infimal length from shrinking to a point. However, we still need to be concerned that γ_0 might not be in $[\gamma]$. After all, knotted regions can pull tight when decreasing length as seen in Fig. 3. To prevent this, we fix scale in a carefully chosen way. In [6] Kusner and Sullivan define the b -distortion thickness of a curve γ as

$$\tau_b(\gamma) = \inf_{dq_\gamma(p, q) \geq b} \|\gamma(p) - \gamma(q)\|,$$

the infimal distance between pairs of points on γ with distortion quotient at least b . We require that $\tau_{\delta([\gamma])}(\gamma) \geq 1$ for curves $\gamma \in U_C([\gamma])$.

We are now prepared to state the main theorem. For now we think of a “ $\delta([\gamma])$ -drc”, which stands for $\delta([\gamma])$ distortion realizing chord, as a pair of points (s, t) for which $dq_\gamma(s, t) = \delta([\gamma])$. We will define this term precisely in Definition 3.

Theorem 1 (Main theorem). *Let $U_C([\gamma])$ be the set of all finite total curvature curves γ in $[\gamma]$, with distortion $\delta(\gamma) < C$ and distortion thickness $\tau_{\delta([\gamma])} \geq 1$.*

¹ This was first pointed out in print by O'Hara [10], who observed that the distortion of a number of tiny knots arranged around a large circle is independent of the number of knots.

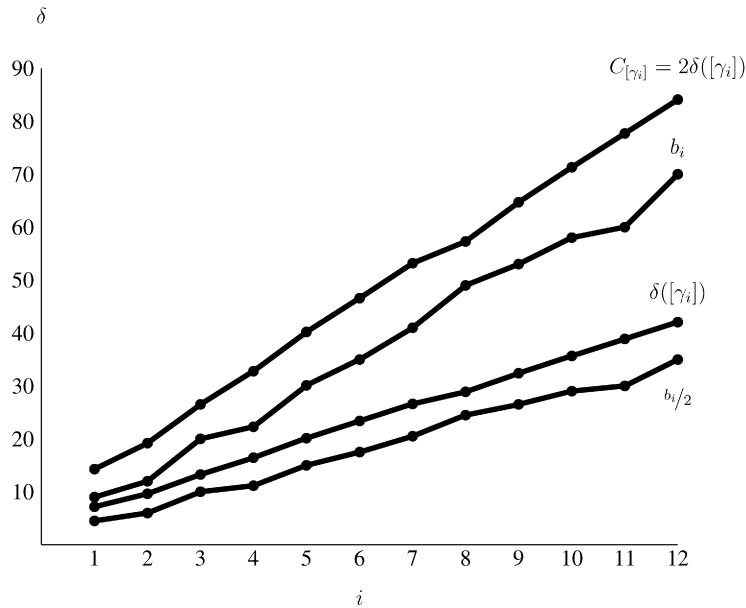


Fig. 2. Here we see a graph representing a divergent sequence $\{b_i\}$ of lower bounds on the distortion of certain curves in $U_{C_{\gamma_i}}([\gamma_i])$. If we define $C_{\gamma_i} = 2\delta([\gamma_i])$ then this will also yield a divergent sequence $\{b_i/2\}$ of lower bounds on $\delta([\gamma_i])$.

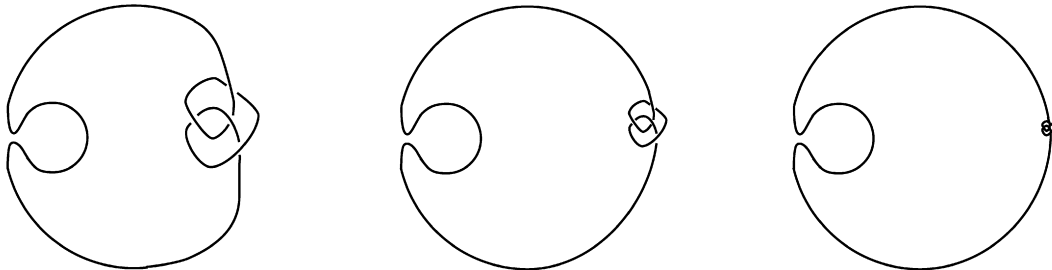


Fig. 3. Here are three representatives from a sequence of figure eight knots with decreasing length that converges to the unknot. The pinch on the left illustrates the concern that every member of the sequence of curves may have maximum distortion realized by a pair of points a constant distance apart even though the knotted region is shrinking to a point.

Then any open interval on a curve of minimum length in $U_C([\gamma])$ is either a straight line segment or contains an endpoint of a $\delta([\gamma])$ -drc.

The proof of the main theorem will be a proof by contradiction. Suppose that we have a curve $\gamma : [a, b] \rightarrow \mathbb{R}^3$ of minimum length in $U_C([\gamma])$ and an arc $\gamma((c, d))$ on γ with positive total curvature that does not contain an endpoint of a $\delta([\gamma])$ -drc. We show that it is possible to decrease the length of the arc $\gamma((c, d))$ to obtain a new curve that is also a member of $U_C([\gamma])$. We have then reached a contradiction since γ is a curve of minimum length in $U_C([\gamma])$. The proof requires two propositions.

- (1) There exists $\varepsilon > 0$ so that $dq(s, t) < \delta([\gamma]) - \varepsilon$ for all points $(s, t) \in (c, d) \times [a, b] \cup [a, b] \times (c, d)$ (Proposition 1).
- (2) The arc $\gamma((c, d))$ can be changed in a length decreasing way so that the increase of the distortion quotient on $(c, d) \times [a, b] \cup [a, b] \times (c, d)$ is less than ε and the change of the distortion quotient on the remainder of $[a, b] \times [a, b]$ is nonpositive. Hence the distortion of γ is not increased (Proposition 2).

We point out that if dq_γ could be extended to a continuous function on $[a, b] \times [a, b]$ then (1) would be immediate and (2) would be much easier to prove. However, this is not always possible for finite total curvature curves as we will see in Section 3.

2. Definitions and background

The proof of our main theorem makes use of several standard results from analysis as well as few results more directly related to distortion. We will be dealing with the set of curves with finite total curvature in the sense of Milnor in [7]. Given a polygon p with vertices v_0, \dots, v_n , we can define a sequence of angles $\alpha_0, \dots, \alpha_n$ where α_i is the exterior angle between the two edges incident to v_i . In [7], Milnor defines the total curvature of the polygon p to be $\kappa_M(p) := \sum_{i=0}^n \alpha_i$. He then defines the total curvature of a curve γ to be $\sup_{p \in \text{Pol}(\gamma)} \kappa_M(p)$, where $\text{Pol}(\gamma)$ denotes the set of all polygons inscribed in γ . It is also shown in [7] that if $\gamma \in C^2$, then $\kappa_M(\gamma)$ agrees with the standard definition for total curvature. There are many interesting properties of the class of finite total curvature curves, denoted FTC, which are explored in detail in both [11] and [9]. In particular, if $\gamma \in \text{FTC}$, then γ has one-sided derivatives everywhere. Therefore we can define both the left and right tangent indicatrices $T_\gamma^\pm : [a, b] \rightarrow S^2$ by

$$T_\gamma^\pm(s) := \lim_{x \rightarrow s^\pm} \frac{\gamma(x) - \gamma(s)}{\|\gamma(x) - \gamma(s)\|}.$$

It can be shown that each of these curves is of bounded variation and therefore, by the structure theorem for BV functions, they possess a derivative in the sense of the norm of their derivative is a Radon measure \mathcal{K} . We find an explicit construction of this measure \mathcal{K} in [9]. The measure theoretic properties of \mathcal{K} which are required for the proof are given in Lemma 4 whose proof we move to Appendix A.

The distortion of a curve increases as points on the curve become closer together in space while remaining relatively far apart in arclength. For arcs of curves that are nearly straight the value of the distortion quotient for pairs on this arc must remain fairly close to 1. Indeed Denne and Sullivan show in [3] the following relationship between total curvature and distortion.

Lemma 1 (Denne/Sullivan). *Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be any finite total curvature curve and let $\gamma([c, d])$ be any arc of γ with total curvature $\kappa \leq \pi$. Then $dq_\gamma(s, t) \leq \sec \kappa / 2$ for all $(s, t) \in [c, d] \times [c, d]$.*

We omit the proof of Lemma 1 which appears in [3] and [9].

3. On the discontinuity of dq_γ for $\gamma \in \text{FTC}$

Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a continuous embedded curve. Then since γ is continuous, the functions $d(\cdot, \cdot; \gamma)$ and $d(\cdot, \cdot; \mathbb{R}^3)$, restricted to points on γ , are each continuous functions. So their ratio, the distortion quotient, is continuous whenever the denominator is nonzero. Indeed, dq_γ is only defined for points off the diagonal of $[a, b] \times [a, b]$.

Regrettably, if $\gamma \in \text{FTC}$ then it may not be possible to define dq_γ on the diagonal in a way that will result in a continuous function. While curves in FTC have one-sided tangents everywhere, the right and left tangent may not be equal at a given point s_0 . This can cause a non-removable discontinuity of dq_γ at (s_0, s_0) . The following example illustrates such a scenario.

Let $\mathcal{C} : [a, b] \rightarrow \mathbb{R}^2$ denote the comet shaped curve in Fig. 4 consisting of an arc A of a circle and a polygonal section, consisting of two line segments P_1 and P_2 , with exterior angle ϕ . Let $c \in [a, b]$ so that $\mathcal{C}(c)$ is the corner point. A calculation shows that $\delta(\mathcal{C}) = \sec(\phi/2)$ and $dq_{\mathcal{C}}(s, t) = \delta(\mathcal{C})$ on the continuum of points

$$E = \{(s, t) \in [a, b] \times [a, b] : \mathcal{C}(s) \in P_1, \mathcal{C}(t) \in P_2, \text{ and } d(\mathcal{C}(s), \mathcal{C}(c), \mathcal{C}) = d(\mathcal{C}(t), \mathcal{C}(c), \mathcal{C})\}.$$

Now, if $\{(s_i, t_i)\} \subset E$ is a sequence so that $(s_i, t_i) \rightarrow (c, c)$ as $i \rightarrow \infty$, then

$$\lim_{i \rightarrow \infty} dq_{\mathcal{C}}(s_i, t_i) = \sec \frac{\phi}{2}.$$

On the other hand, for all points s so that $\mathcal{C}(s)$ is a point on either one of the line segments, we have $dq_{\mathcal{C}}(s, c) = 1$, hence

$$\lim_{s \rightarrow c} dq_{\mathcal{C}}(s, c) = 1.$$

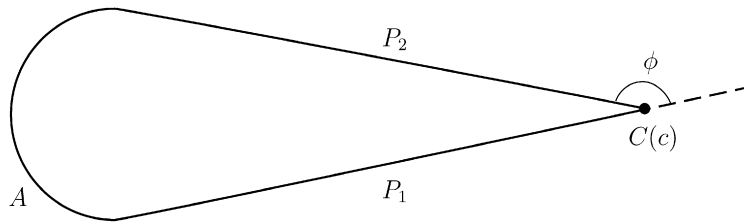


Fig. 4. This example illustrates a discontinuity of the distortion quotient. The discontinuity of the tangent curve at the point corresponding to the corner point causes a jump in the distortion quotient.

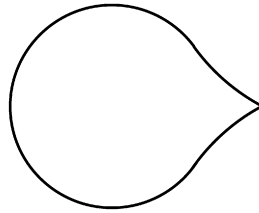


Fig. 5. The distortion of this curve is realized as a limit of distortion quotients of pairs of symmetric points converging to the corner point. The parameter space $[a, b] \times [a, b]$ has no point (s, t) that realizes the distortion.

We can readily see that the discontinuity along the diagonal of $[a, b] \times [a, b]$ is not removable. In fact, since the distortion is defined as a supremum, it may be the case that for a general curve $\gamma \in \text{FTC}$, there exists a sequence of points (s_i, t_i) converging to a point (s, s) so that $dq_\gamma(s_i, t_i) \rightarrow \delta(\gamma)$, $dq_\gamma(s, t) < \delta(\gamma)$ everywhere dq_γ is defined, and dq_γ has a non-removable discontinuity at (s, s) . Curves of this type have no distortion realizing chord. An example of such a curve is the *Dragon's tooth curve* in Fig. 5.

This curve, $\mathcal{D}: [a, b] \rightarrow \mathbb{R}^2$, is formed by three circle arcs: one of small radius r and two of larger radius R . The large circle arcs are connected to the smaller circle arc so that their tangents agree at the points of intersection. The large circles meet at a corner with exterior angle $\phi < \pi$. The distortion of a circle is $\pi/2$, so if $\mathcal{D}(s)$ and $\mathcal{D}(t)$ are points on the same circle arc then $dq_{\mathcal{D}}(s, t) \leq \pi/2$. If $\mathcal{D}(s)$ lies on one of the large circle arcs and if $\mathcal{D}(t)$ lies on the small circle arc, then $dq_{\mathcal{D}}(s, t) \rightarrow 1$ as $d(\mathcal{D}(s), \mathcal{D}(t); \mathbb{R}^2) \rightarrow 0$. Therefore, the distortion quotient cannot grow large because the denominator becomes small. On the other hand, the numerator is no larger than the length of the two arcs. So $dq_{\mathcal{D}}(s, t)$ is uniformly bounded above for all points (s, t) for which $\mathcal{D}(s)$ and $\mathcal{D}(t)$ lie on circle arcs of different radius.

Let $\mathcal{D}(c)$ denote the corner point. Let $\{(s_i, t_i)\}$ be a sequence so that $\mathcal{D}(s_i)$ is on one large circle arc, $\mathcal{D}(t_i)$ is on the other large circle arc and $d(\mathcal{D}(s_i), \mathcal{D}(c); \mathcal{D}) = d(\mathcal{D}(t_i), \mathcal{D}(c); \mathcal{D})$ for all i . If $(s_i, t_i) \rightarrow (c, c)$ then

$$\lim_{i \rightarrow \infty} dq_{\mathcal{D}}(s_i, t_i) = \sec \frac{\phi}{2}.$$

Forcing ϕ to be close to π in the construction of \mathcal{D} forces $dq_{\mathcal{D}}(s, t) < \sec \phi/2$ for all (s, t) in the domain of $dq_{\mathcal{D}}$. So, the distortion of \mathcal{D} is evaluated as a limit of $dq_{\mathcal{D}}(s, t)$ as (s, t) converges to a point on the diagonal that is a non-removable discontinuity.

To deal with examples like this, the definition of a distortion realizing chord will need to be extended. Just considering pairs of points (s, t) so that $dq_\gamma(s, t) = \delta(\gamma)$ is not sufficient. After all, there may be no such pairs. We start with a definition.

Definition 1. For any $\gamma: [a, b] \rightarrow \mathbb{R}^3$ in FTC, we define the function $D_\gamma: [a, b] \rightarrow \mathbb{R}$ by

$$D_\gamma(s) := \sup_t dq_\gamma(s, t).$$

Lemma 2. If $\gamma: [a, b] \rightarrow \mathbb{R}^3$ is a curve with finite total curvature parametrized by arclength, then

$$D_\gamma(s) = \sup_t \frac{d(\gamma(s), \gamma(t); \gamma)}{d(\gamma(s), \gamma(t); \mathbb{R}^3)} = \max_t \frac{d(\gamma(s), \gamma(t); \gamma)}{d(\gamma(s), \gamma(t); \mathbb{R}^3)}.$$

Proof. It can be shown that one-sided derivatives exist everywhere for curves in FTC (see [9]), hence

$$\lim_{t \rightarrow s^+} \frac{\gamma(t) - \gamma(s)}{t - s}$$

exists. We know, by the continuity of the norm function $\|\cdot\|$, that

$$\lim_{t \rightarrow s^+} \left\| \frac{\gamma(t) - \gamma(s)}{t - s} \right\| = \lim_{t \rightarrow s^+} \frac{\|\gamma(t) - \gamma(s)\|}{|t - s|}$$

exists and equals 1 since the curve is arclength parametrized. So,

$$\lim_{t \rightarrow s^+} \frac{|t - s|}{\|\gamma(t) - \gamma(s)\|} = 1.$$

But this is exactly the one-sided limit of the distortion quotient function $dq_\gamma(s, t)$ if we leave s fixed. A similar computation shows that the limit from the left exists as well and is also equal to 1. Therefore, since the function $dq_\gamma(s, t)$ with s fixed is a continuous function on the compact set $[a, b]$, it follows that it achieves its maximum at a point in $[a, b]$. \square

In the case of the Dragon's tooth \mathcal{D} , the function $D_{\mathcal{D}}(s)$ is defined at the corner point, but it is less than the distortion of the curve. However, we do have $D_{\mathcal{D}}(s_i) \rightarrow \delta(\mathcal{D})$ as $s_i \rightarrow s$. We will replace D_γ so that our new function will achieve its supremal value on γ .

Definition 2. If $\gamma : [a, b] \rightarrow \mathbb{R}^3$ is a curve with finite total curvature parametrized by arclength, then we define the *distortion shadow*, denoted $\overline{D}_\gamma(s)$, to be the upper envelope of $D_\gamma(s)$. That is

$$\overline{D}_\gamma(s) := \inf_{\varepsilon > 0} \sup_{|s-x| < \varepsilon} \left(\max_t \frac{d(\gamma(x), \gamma(t); \gamma)}{d(\gamma(x), \gamma(t); \mathbb{R}^3)} \right).$$

We are now in a position to make the definition of a distortion realizing chord precise. The Dragon's tooth example shows that we will need to define the distortion quotient along the diagonal in a way that guarantees that the value of $dq_\gamma(s, s)$ is defined to be the largest limiting value of all sequences approaching it. For this we again use the upper envelope.

Definition 3. Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a finite total curvature curve parametrized by arclength. If (s, t) is such that

$$\overline{dq}_\gamma(s, t) := \inf_{\varepsilon > 0} \sup_{\|(x, y) - (s, t)\| < \varepsilon} \frac{d(\gamma(x), \gamma(y); \gamma)}{d(\gamma(x), \gamma(y); \mathbb{R}^3)} \geq k,$$

then we say (s, t) (or the chord with endpoints $\gamma(s)$ and $\gamma(t)$) is a *k-distortion realizing chord (k-drc)*.

Lemma 3. Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a finite total curvature curve parametrized by arclength. Then $\overline{D}_\gamma(s) \geq k$ if and only if there exists a value t so that the chord with endpoints $\gamma(s)$ and $\gamma(t)$ is a *k-drc*.

Proof. First assume that there is a value t so that the point (s, t) defines a *k-drc*. We know that there exists a sequence $\{(s_i, t_i)\}$ so that $(s_i, t_i) \rightarrow (s, t)$ and $dq_\gamma(s_i, t_i) \rightarrow K \geq k$. Since the function $D_\gamma(s_i)$ computes the maximum over all values of t , it is evident that $D_\gamma(s_i) \geq dq_\gamma(s_i, t_i)$. Furthermore, since the upper envelope is upper-semicontinuous, $\lim \overline{D}_\gamma(s_i) \geq \lim D_\gamma(s_i)$. So our string of inequalities then becomes

$$\overline{D}_\gamma(s) \geq \lim \overline{D}_\gamma(s_i) \geq \lim D_\gamma(s_i) \geq \lim dq_\gamma(s_i, t_i) = K \geq k.$$

The reverse implication is immediate from the definitions. Let $\{s_i\} \subset [a, b]$ be a sequence so that $D_\gamma(s_i) \rightarrow K \geq k$. For each s_i there exists a t_i so that $D(s_i) = dq_\gamma(s_i, t_i)$ by Lemma 2. Furthermore, since the points (s_i, t_i) are elements of the compact set $[a, b] \times [a, b]$ we may assume, by restricting to a subsequence, that $(s_i, t_i) \rightarrow (s, t)$ for some $(s, t) \in [a, b] \times [a, b]$. It remains to show $\overline{dq}_\gamma(s, t) \geq k$. Indeed, for any $\varepsilon > 0$

$$\sup_{\|(x, y) - (s, t)\| < \varepsilon} dq(x, y) \geq k$$

since the set on which the supremum is taken contains infinitely many elements of the sequence $\{(s_i, t_i)\}$. Therefore,

$$\overline{dq}_\gamma(s, t) = \inf_{\varepsilon > 0} \sup_{\|(x, y) - (s, t)\| < \varepsilon} dq(s, t) \geq k.$$

So the pair (s, t) defines a k -drc. \square

By defining k -drc's in terms of an upper-semicontinuous function we have made the proof of the first step towards the main theorem relatively simple. Indeed, upper-semicontinuous functions achieve their maximum value on compact sets. Therefore, if a closed arc contains no endpoints of k -drcs, then the maximum value of \overline{dq}_γ is bounded away from k by a positive quantity ε . Hence, the distortion quotient is bounded away from k by a positive quantity. We make this rigorous below.

Proposition 1. *Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a curve of finite total curvature. If $(c, d) \subset [a, b]$ so that $\overline{dq}_\gamma(s, t) < k$ for all $(s, t) \in (c, d) \times [a, b] \cup [a, b] \times (c, d)$ then, for any subinterval $[p, q] \subset (c, d)$ there exists an $\varepsilon > 0$ so that $dq_\gamma(s, t) \leq k - \varepsilon$ for all $(s, t) \in (p, q) \times [a, b] \cup [a, b] \times (p, q)$ with $s \neq t$.*

Proof. By assumption the interval (c, d) is free from endpoints of k -drc's. Then using Lemma 3, it follows that for every $s \in (c, d)$ the value of $\overline{D}_\gamma(s) < k$. Let $[p, q]$ be any closed interval subset of (c, d) . Since $\overline{D}_\gamma(s)$ is upper-semicontinuous and bounded, $\overline{D}_\gamma(s)$ has a maximum M on $[p, q]$. We can let $\varepsilon = k - M$. \square

4. Decreasing length without increasing distortion

Now that we know that arcs of γ which do not contain k -drc's have distortion bounded away from $\delta(\gamma)$ by some positive quantity ε , it remains to show the length of an arc can be decreased while changing the distortion quotient by an amount smaller than ε . This will complete the second step in the proof of the Main theorem outlined in Section 2.1.

Proposition 2. *Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a finite total curvature curve and suppose $\gamma((c, d))$ is any arc with nonzero total curvature. Then, given $\varepsilon > 0$, we can replace $\gamma((c, d))$ with an arc $P((c, d))$ of shorter length so that $dq_P(s, t) - dq_\gamma(s, t) < \varepsilon$ for all $(s, t) \in (c, d) \times [a, b] \cup [a, b] \times (c, d)$ with $s \neq t$.*

Proof. Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a curve parametrized by arclength and suppose that $\gamma((c, d))$ is an arc of γ of length $L = d - c$ with nonzero total curvature. Let $\varepsilon > 0$ be given. This proof has several cases. We will first assume that we can find a subarc of $\gamma((c, d))$ whose total curvature is small. Without loss of generality we assume this is the arc $\gamma((c, d))$. We will then modify only a subarc $\gamma([\hat{c}, \hat{d}])$ of $\gamma((c, d))$. To prove that distortion has not greatly increased, we will then examine the change in the distortion quotient. Pairs of points s and t which are relatively close will require a different argument than pairs where s and t are relatively far apart. Fig. 6 may help the reader to understand the different components of the proof.

Case 1. Assume the arc $\gamma((c, d))$ has total curvature $0 < K < 2 \cos^{-1}(1/(1 + \varepsilon/2))$; then by Lemma 1, the distortion of the arc $\gamma((c, d))$ is no more than $1 + \varepsilon/2$.

Proof. By replacing the interval $\gamma([c, d])$ with an inscribed polygon, we can decrease length and not increase total curvature. By Lemma 1, since the total curvature has not increased, the distortion of the arc remains between 1 and $1 + \varepsilon/2$. Hence the change in distortion of the arc is smaller than ε . This takes care of all points in region C in Fig. 6.

Now let us turn our attention to a subarc $\gamma([\hat{c}, \hat{d}]) \subset \gamma((c, d))$. We define $[\hat{c}, \hat{d}]$ to be any interval contained in (c, d) so that $\gamma([\hat{c}, \hat{d}])$ has positive curvature. There exists a Radon measure \mathcal{K} on $[a, b]$ so that $\mathcal{K}((p, q))$ is the total curvature of $\gamma((p, q))$ for all intervals $(p, q) \subset (a, b)$. We will use \mathcal{K} to show that we can find an interval $[\hat{c}, \hat{d}]$ with the desired properties. Since \mathcal{K} is a Radon measure, if we let H denote the set of all compact subsets of (c, d) then

$$\mathcal{K}((c, d)) = \sup_{h \in H} \mathcal{K}(h)$$

Hence there must exist some sequence of compact sets h_i so that $\mathcal{K}(h_i) \rightarrow \mathcal{K}((c, d))$ as $i \rightarrow \infty$ and so there must also be some j so that $\mathcal{K}(h_j) > 0$. Let $[\hat{c}, \hat{d}] \subset (c, d)$ be any closed interval containing h_j . Notice that we can replace

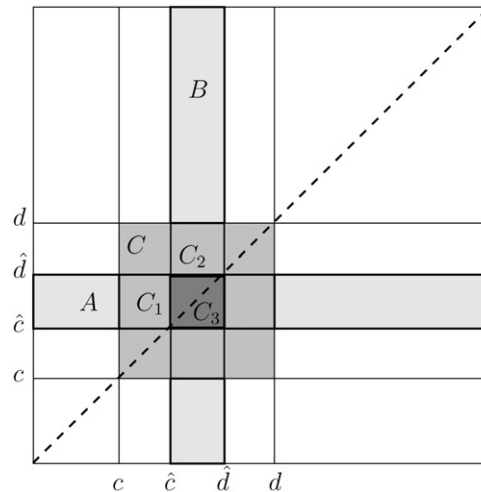


Fig. 6. We break up our proof into several pieces according to a subdivision of the parameter space. Define $A := [a, c] \times [\hat{c}, \hat{d}]$, $B := [\hat{c}, \hat{d}] \times [d, b]$, $C := [c, d] \times [c, d]$, $C_1 := [c, \hat{c}] \times [\hat{c}, \hat{d}]$, $C_2 := [\hat{c}, \hat{d}] \times [\hat{d}, d]$, and $C_3 := [\hat{c}, \hat{d}] \times [\hat{c}, \hat{d}]$. We use Lemma 1 to control the change in the distortion quotient inside the region C . Then, we restrict our attention to the regions A and B , which are bounded away from the diagonal of $[a, b] \times [a, b]$, and use a more straightforward calculation to show that the change in the distortion quotient can be controlled on regions A and B .

$\gamma[\hat{c}, \hat{d}]$ with a polygon without increasing the distortion too much for points in C_1 , C_2 , and C_3 in Fig. 6 since each of these regions is contained within the larger region C .

For any given $\varepsilon' > 0$, let $P: [a, b] \rightarrow \mathbb{R}^3$ be a curve that consists of two types of arcs. The first is any polygon inscribed inside the arc $\gamma([\hat{c}, \hat{d}])$ such that $\|\gamma(s) - P(s)\| < \varepsilon'$ for all $s \in [\hat{c}, \hat{d}]$, $P(\hat{c}) = \gamma(\hat{c})$, and $P(\hat{d}) = \gamma(\hat{d})$. The second arc satisfies the relation $P(s) = \gamma(s)$ for all $s \in [a, b] \setminus [\hat{c}, \hat{d}]$. Even if the arc $\gamma([\hat{c}, \hat{d}])$ is already a polygon, we can replace $\gamma([\hat{c}, \hat{d}])$ with $P([\hat{c}, \hat{d}])$ so as to decrease length.

It remains to show that we can choose ε' small enough on $[\hat{c}, \hat{d}]$ so that

$$dq_P(s, t) - dq_\gamma(s, t) < \varepsilon \quad \text{for all pairs } (s, t) \in A \cup B,$$

where A and B are defined in Fig. 6.

Notice that if $dq_P(s, t) < dq_\gamma(s, t)$ then there is nothing to show, since we are only concerned with *increasing* distortion. So we will assume that $dq_P(s, t) > dq_\gamma(s, t)$. Suppose that $(s, t) \in A$. Then, using the definition of distortion quotient,

$$\begin{aligned} dq_P(s, t) - dq_\gamma(s, t) &= \frac{d(\gamma(s), \gamma(t); \mathbb{R}^3)d(P(s), P(t); P) - d(P(s), P(t); \mathbb{R}^3)d(\gamma(s), \gamma(t); \gamma)}{d(P(s), P(t); \mathbb{R}^3)d(\gamma(s), \gamma(t); \mathbb{R}^3)}. \end{aligned}$$

But $(s, t) \in A$ and since A is bounded away from the diagonal of $[a, b] \times [a, b]$, we know that s and t must be at least $\min\{\hat{c} - c, d - \hat{d}\} > 0$ apart. Thus, since the curves γ and P are both embeddings, we know there is a constant C so that

$$C \geq \frac{1}{d(P(s), P(t); \mathbb{R}^3)d(\gamma(s), \gamma(t); \mathbb{R}^3)}$$

for all $(s, t) \in A$. A calculation shows that

$$\frac{dq_P(s, t) - dq_\gamma(s, t)}{C} \leq d(\gamma(s), \gamma(t); \gamma)[d(\gamma(s), \gamma(t); \mathbb{R}^3) - d(P(s), P(t); \mathbb{R}^3)].$$

As

$$-d(P(s), P(t); \mathbb{R}^3) \leq d(P(s), \gamma(s); \mathbb{R}^3) - d(P(t), \gamma(s); \mathbb{R}^3)$$

and

$$d(\gamma(s), \gamma(t); \mathbb{R}^3) \leq d(P(t), \gamma(s); \mathbb{R}^3) + d(P(t), \gamma(t); \mathbb{R}^3),$$

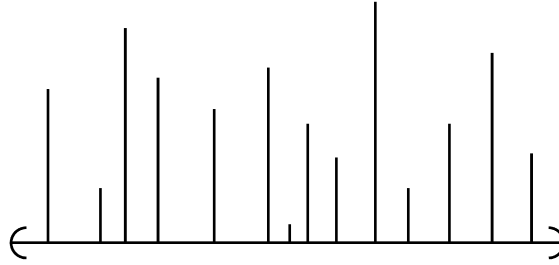


Fig. 7. Here we see a partition of the interval (c, d) where the points that define the partition are the first M atoms in the sequence $\{x_i\}$.

we see

$$d(\gamma(s), \gamma(t); \mathbb{R}^3) - d(P(s), P(t); \mathbb{R}^3) \leq d(P(t), \gamma(t); \mathbb{R}^3) + d(P(s), \gamma(s); \mathbb{R}^3) \leq 2\varepsilon'.$$

Finally, since $s, t \in [\hat{c}, \hat{d}]$, we can choose ε' so that

$$2\varepsilon' < \varepsilon / (Cd(\gamma(s), \gamma(t); \gamma)),$$

completing the proof in this case. \square

Case 2. Suppose there is no subarc of $\gamma((c, d))$ with total curvature K where

$$0 < K < 2\cos^{-1}(1/(1 + \varepsilon/2)).$$

Proof. We will show that this forces $\gamma((c, d))$ to be a polygonal arc. A new technique will be required in this case.

By the Lebesgue–Radon–Nikodym theorem, we can write the curvature measure \mathcal{K} as a sum of two measures λ_a and λ_s so that λ_a is absolutely continuous with respect to ds and λ_s is singular with respect to ds . We then further decompose λ_s as a sum of atomic measures $\{\mu_{x_i}\}$ and a non-atomic measure μ which is singular with respect to arclength. So we have

$$\mathcal{K}((p, q)) = \lambda_a((p, q)) + \sum_i \mu_{x_i}((p, q)) + \mu((p, q)).$$

Suppose now that $K_0 := 2\cos^{-1}(1/(1 + \varepsilon/2))$ and (p, q) is an interval so that $\mathcal{K}((p, q)) > K_0$. Then $\sum_i \mu_{x_i}((p, q)) < \infty$ and so there exists a constant M so that $\sum_{i>M} \mu_{x_i}((p, q)) < K_0/2$. Without loss of generality, assume $x_1 < \dots < x_M$. Then we partition the interval (p, q) with the points $\{x_1, x_2, \dots, x_M\}$ as seen in Fig. 7. \square

We now make use of the following property of non-atomic Radon measures.

Lemma 4. Suppose that μ is a non-atomic Radon measure (i.e., $\mu(\{p\}) = 0$ for each $p \in \mathbb{R}$) defined on a finite interval (a, b) . Then we can find a value $L \in \mathbb{R}$ so that the measure of every subinterval (c, d) of (a, b) with length less than L satisfies the inequality $\mu((c, d)) \leq 2\mu((a, b))/3$.

Now, let (u, v) be a subinterval of (p, q) that lies between two adjacent partition points. Since the measure $\lambda_a + \mu$ is non-atomic, by (possibly repeated) application of Lemma 4 there is a value L so that if $|v - u| < L$, then $\lambda_a((u, v)) + \mu((u, v)) < K_0/2$. Thus, we may assume that (u, v) has small enough length so that $\lambda_a((u, v)) + \mu((u, v)) < K_0/2$. Since (u, v) does not include any of the partition points x_1, \dots, x_M ,

$$\sum_i \mu_{x_i}((u, v)) = \sum_{i>M} \mu_{x_i}((u, v)) < K_0/2.$$

Thus $\mathcal{K}((u, v)) < K_0$ and it must be the case that $\mathcal{K}((u, v)) = 0$. Covering each interval between the M partition points $\{x_1, x_2, \dots, x_M\}$ by overlapping open intervals with length less than L shows us that $\mathcal{K}((p, q) - \{x_1, \dots, x_M\}) = 0$. Therefore, the curve is polygonal on the interval (p, q) with corners at the atoms $\{x_1, x_2, \dots, x_M\}$. Since this is true for any interval (p, q) it must be the case that the arc $\gamma((c, d))$ is polygonal.

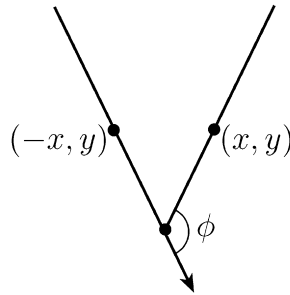


Fig. 8. If we translate the curve so that the vertex at the corner of the arc is symmetric about the y -axis, then a calculation shows that the distortion quotient restricted to this pair of segments is maximized at points of the form (x, y) and $(-x, y)$. Moreover, this maximum is exactly equal to $\sec \frac{\phi}{2}$ where ϕ is the exterior angle at the corner. See [9] for a proof.

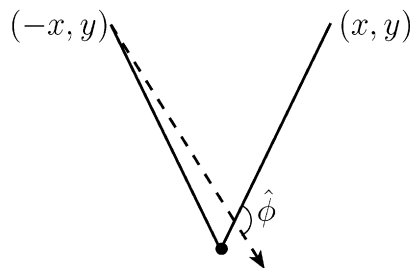


Fig. 9. Here we shorten the length of the polygonal arc slightly while decreasing the exterior angle by a small amount by shortening one segment. This modification decreases the distortion quotient on this pair of line segments.

It remains to describe how to decrease length without changing the distortion quotient any more than ε . First we will restrict our attention to a neighborhood $\gamma((p, q))$ of one corner small enough to guarantee that no pair of points in the neighborhood are $\text{Len}(\gamma)/2$ apart along the curve γ . If the exterior angle at the corner point is ϕ , then a calculation in [9] shows the restriction of the distortion quotient to the edges meeting at the corner achieves a maximum value of $\sec \frac{\phi}{2}$ as seen in Fig. 8.

Let $\gamma(\hat{p})$ and $\gamma(\hat{q})$ be the points corresponding to the points labeled $(-x, y)$ and (x, y) , respectively, in Fig. 8. Without loss of generality, assume $d(\gamma(p), \gamma(q); \gamma) > d(\gamma(\hat{p}), \gamma(\hat{q}); \gamma)$. As seen in Fig. 9, we replace the edge with vertices $(0, 0)$ and (x, y) with an edge with vertices $\varepsilon(x, y)$ and (x, y) (using a small value of ε) and replace the edge with vertices $(-x, y)$ and $(0, 0)$ with the edge with vertices $(-x, y)$ and $\varepsilon(x, y)$. As before, for given $\varepsilon' > 0$ define the new curve $P: [a, b] \rightarrow \mathbb{R}^3$ by $\gamma(s) = P(s)$ for all $s \notin [\hat{p}, \hat{q}]$, $P(\hat{p}) = \gamma(\hat{p})$, $P(\hat{q}) = \gamma(\hat{q})$, and $P([\hat{p}, \hat{q}])$ is the modified corner described above so that $\|\gamma(s) - P(s)\| < \varepsilon'$ for all $s \in [\hat{p}, \hat{q}]$. The triangle inequality guarantees that this alteration will decrease length. Since $\sec(\phi/2)$ is an increasing function on the interval $(0, \pi)$, then decreasing the exterior angle either decreases distortion at points in $[\hat{p}, \hat{q}] \times [\hat{p}, \hat{q}]$ or remains constant. By decreasing the value ε used to chop off the corner we make the new exterior angle at $P(\hat{p})$ arbitrarily close to zero, hence the distortion quotient evaluated at points nearby remain close to 1. If $s, t \in (p, q) \times (p, q)$ so that $P(s)$ and $P(t)$ straddle both corners in $P((p, q))$, then $dq_P(s, t) - dq_\gamma(s, t) < 0$ since we have decreased the arclength between the points while their position in space remains unchanged. Finally, the argument used to control distortion on the sets A and B from Case 1 can be used to show that the distortion at analogous points in $[\hat{p}, \hat{q}] \times ([a, b] \setminus [\hat{p}, \hat{q}])$ and $([a, b] \setminus [\hat{p}, \hat{q}]) \times [\hat{p}, \hat{q}]$ has changed by no more than ε .

This completes the proof. \square

5. Main theorem

We now state the main theorem. Recall that distortion thickness $\tau_{\delta([\gamma])}(\gamma) \geq 1$ means that for any pair (s, t) with $dq_\gamma(s, t) \geq \delta([\gamma])$ we have $\|\gamma(s) - \gamma(t)\| \geq 1$.

Theorem 1 (Main theorem). Let $U_C([\gamma])$ be the set of all finite total curvature curves γ in $[\gamma]$, with distortion $\delta(\gamma) < C$ and distortion thickness $\tau_{\delta([\gamma])} \geq 1$.

Then any open interval on a curve of minimum length in $U_C([\gamma])$ is either a straight line segment or contains an endpoint of a $\delta([\gamma])$ -drc.

Proof. Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a curve of minimum length in $U_C([\gamma])$ and let (c, d) be any open interval on γ . If the total curvature along (c, d) is zero, then there is nothing to show. So assume that the total curvature along (c, d) is positive. Further assume that the interval (c, d) does not contain an endpoint of a $\delta([\gamma])$ -drc. We will deduce a contradiction. Since the total curvature along (c, d) is nonzero, there exists a subinterval (p, q) of (c, d) such that the total curvature along (p, q) is nonzero. Since there is no endpoint of a $\delta([\gamma])$ -drc contained within (c, d) , Proposition 1 states that if $(s, t) \in (p, q) \times [a, b]$, then $\delta([\gamma]) - dq_\gamma(s, t) > \varepsilon$ for some $\varepsilon > 0$. Furthermore Proposition 2 enables us to decrease the length of the interval (p, q) to obtain a new curve $\hat{\gamma}$ with the property that $dq_{\hat{\gamma}}(s, t) - dq_\gamma(s, t) < \varepsilon$ for $(s, t) \in (c, d) \times [a, b] \cup [a, b] \times (c, d)$. Thus no new $\delta([\gamma])$ -drc's exist with endpoints in $(c, d) \times [a, b] \cup [a, b] \times (c, d)$. We claim that we have not increased the distortion quotient at any pair outside $(c, d) \times [a, b] \cup [a, b] \times (c, d)$. Indeed, if (s, t) is any point outside $(c, d) \times [a, b] \cup [a, b] \times (c, d)$, then $d(\hat{\gamma}(s), \hat{\gamma}(t); \hat{\gamma}) \leq d(\gamma(s), \gamma(t); \gamma)$ while $d(\hat{\gamma}(s), \hat{\gamma}(t); \mathbb{R}^3) = d(\gamma(s), \gamma(t); \mathbb{R}^3)$. Since $\delta(\gamma) \geq \delta([\gamma])$, the distortion of γ is realized on such a pair and this implies that $\delta(\hat{\gamma}) \leq \delta(\gamma) < C$. Therefore, $\hat{\gamma} \in U_C([\gamma])$ and $\text{Len}(\hat{\gamma}) < \text{Len}(\gamma)$. This is the desired contradiction. \square

6. Conclusion and future directions

The main theorem provides a good deal of structure for distortion minimizing curves provided they have a representative of shortest length. Indeed, the frequency of drcs is highly reminiscent of the results in [1]. We hope this new information on drcs will aid in constructing better lower bounds for distortion.

We point out that the main theorem is still somewhat unsatisfactory since we have not yet proved that there are length-minimizing curves in $U_C([\gamma])$. Fixing this problem requires proving at least two conjectures:

Conjecture 1. For any curve γ there exists a chord with distinct endpoints $\gamma(s)$ and $\gamma(t)$ so that $dq_\gamma(s, t) \geq \delta([\gamma])$. Hence we can rescale any knot in $[\gamma]$ to have $\delta([\gamma])$ -distortion thickness 1.

This conjecture means that the set $U_C([\gamma])$ will be easy to work with. In this case we further conjecture:

Conjecture 2. The set $U_C([\gamma])$ contains a minimizer for length for any knot type $[\gamma]$ and any $C > \delta([\gamma])$.

As pointed out earlier, we have a path towards an answer to Gromov's question. It remains to find an appropriate topological invariant and lower bound for distortion in terms of this invariant. A potential candidate for the topological invariant is a quantity known as the hull number defined in [2]. The n th-hull of a curve γ is the set of all points x for which any plane passing through x intersects γ in at least $2n$ points (we count a tangential intersection as a double intersection). Then, the hull number for a curve γ is the largest value of n for which the n th-hull is nonempty. We make this a knot invariant by defining the hull number of the knot type $[\gamma]$ to be the infimal hull number of all knots in $[\gamma]$.

The hull number is of particular interest since it has been shown in [5] that the hull number does not increase for the family $\{\gamma, \gamma \# \gamma, \gamma \# \gamma \# \gamma, \dots\}$ and yet it does increase for a family of knot types which seems to be a prime candidate for having no universal upper bound for distortion (the $(n, n-1)$ -torus knots). The minimum distortions of these knots are therefore particularly interesting.

In [8] we will provide approximate locally distortion minimizing curves in these knot types, computed using simulated annealing.

Appendix A

Lemma 4. Suppose that μ is a non-atomic Radon measure (i.e., $\mu(\{p\}) = 0$ for each $p \in \mathbb{R}$) defined on a finite interval (a, b) . Then given any finite interval (a, b) , we can find a value $L \in \mathbb{R}$ so that the measure of every subinterval (c, d) of (a, b) with length less than L satisfies the inequality $\mu((c, d)) \leq 2\mu((a, b))/3$.

Proof. We will proceed by the method of contradiction. We assume that for all $L > 0$ there exists an interval $(c, d) \subset (a, b)$ with $d - c < L$ so that $\mu((c, d)) > 2\mu((a, b))/3$. For each positive integer n , let $L(n) = (b - a)/2^n$, and let s_n be an open interval with length less than $L(n)$ so that $s_n \subset (a, b)$ and $\mu(s_n) > 2\mu((a, b))/3$. Notice that if we define S_n to be the closure of s_n , then $\mu(S_n) = \mu(s_n)$ since μ contains no atoms. Let $C_n = \{C_{ni}\}$ be the closed cover of $[a, b]$ consisting of the sets $C_{ni} = [a + (i - 1)L(n), a + iL(n)]$ for $i = 1, 2, \dots, 2^n$. So C_1 consists of two sets of equal length $C_{11} = [a, (a + b)/2]$ and $C_{12} = [(a + b)/2, b]$.

Claim 1. *There must be some integer i so that $S_i \subset [a, (a + b)/2]$ or $S_i \subset [(a + b)/2, b]$.*

Proof. If there is no such integer, then it follows that $S_i \cap [a, (a + b)/2] \neq \emptyset$ and $S_i \cap [(a + b)/2, b] \neq \emptyset$ for all i . Therefore, $(a + b)/2 \in S_i$ for all i . Since the length of the S_i 's is approaching zero it follows that

$$\left\{ \frac{(a + b)}{2} \right\} = \bigcap_{i=1}^{\infty} S_i.$$

But then since μ is a Radon measure $\mu(\{(a + b)/2\}) > 2\mu((a, b))/3$, which contradicts the assumption that there are no atoms for μ . \square

Therefore, by Claim 1 there exists some S_i so that either $S_i \subset C_{11}$ or $S_i \subset C_{12}$. Hence, either $\mu(C_{11}) > 2\mu((a, b))/3$ or $\mu(C_{12}) > 2\mu((a, b))/3$. Define that set to be T_1 . Observe that $S_j \subset T_1$ for all $j \geq i$ since otherwise either $S_j \subset (a, b) \setminus T_1$ or μ is atomic. By assumption μ is non-atomic and if $S_j \subset (a, b) \setminus T_1$, then the measure of (a, b) will be too large.

The next open cover C_2 , covers T_1 with exactly two sets. Using the claim again and the fact that $\mu((a, b) \setminus T_1) < \mu((a, b))/3$, we know that only one member of C_2 has μ -measure greater than $2\mu((a, b))/3$. Call it T_2 . Continue inductively to generate a sequence of closed intervals $\{T_n\}$ with the following properties:

- (1) $\mu(T_n) > 2\mu((a, b))/3$ for all n ,
- (2) the length of T_n is exactly $(b - a)/2^n$ for all n , and
- (3) $T_{n+1} \subset T_n$ for all $n \geq 1$, hence $\bigcap_{n=1}^{\infty} T_n \neq \emptyset$.

Since the intersection of the T_i 's is nonempty and since the length of the T_i 's is approaching zero, it follows that the intersection of the T_i 's is a single point and that the μ -measure of this point is larger than $2\mu((a, b))/3 \geq 0$ as in Claim 1. This again contradicts the assumption that μ contains no atoms. \square

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